Angular Motion in a Plane

Week 8, Lesson 1

- Angular Displacement
- Angular Velocity
- Angular Acceleration
- Equations for Uniformly Accelerated Angular Motion
- Relations Between Angular and Tangential Quantities
- Centripetal Acceleration
- Centripetal Force

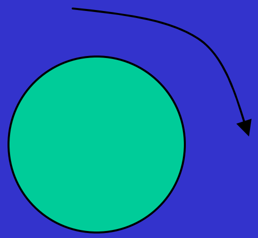
References/Reading Preparation:

Schaum's Outline Ch. 9

Principles of Physics by Beuche – Ch.7

Angular Displacement

To describe the motion of an object on a circular path, or the rotation of a wheel on an axle, we need a coordinate to measure angles.



One revolution of the wheel = 360º

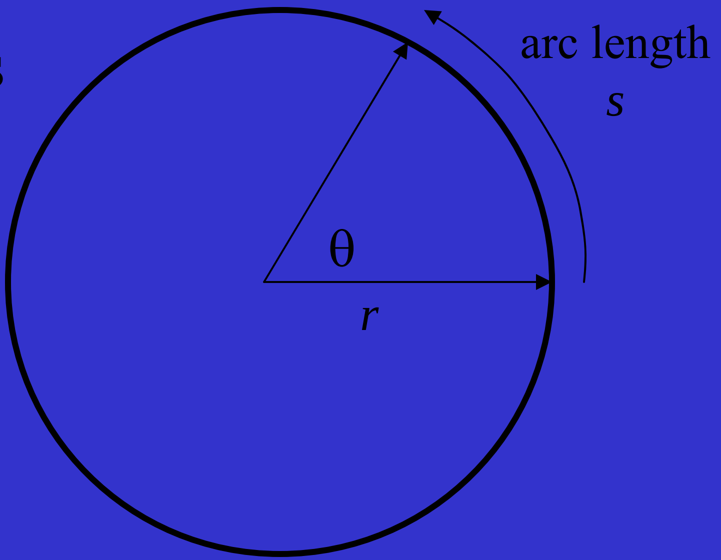
(or, 1 rev = 360º)

Now, ***Angular Displacement*** is expressed in radians.

Where,

One radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

Thus, an angle θ in radians is given in terms of the arc length *s* it subtends on a circle of radius *r* by:



$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r}$$ (θ is in radians)

Now, for $\theta = 360^\circ$, one full circle, $s = 2\pi r$

Therefore, $\theta = 2\pi r/r = 2\pi$

Therefore, $360^\circ = 2\pi$ *radians*

So, one full revolution = $360^\circ = 2\pi$ *radians*

Angular Velocity (how fast something rotates)

The Angular Velocity (ω) of an object is the rate at which its angular coordinate, the angular displacement θ, changes with time.

$$\text{average angular velocity} = \frac{\text{angle turned}}{\text{time taken}}$$

or

$$\overline{\omega} = \frac{\theta}{t}$$

The units of ϖ are typically rad/s, deg/s, or rev/min (rpm)

or

$$\overline{\omega} = \frac{\theta_f - \theta_o}{t}$$

also, $\omega(\text{in rad/s}) = 2\pi f$

where f is the *frequency of rotation* in rev/s

Worked Example

A wheel turns through 1800 rev in 1 min. What is its angular velocity?

$$\bar{\omega} = \frac{\theta}{t} = 1800 \text{ rev}/\ 60 \text{ s}$$

$$= 30 \text{ rev/s}$$

Now, $$\frac{30 \text{ rev}}{\text{s}} = \frac{30 \text{ rev}}{\text{s}} \times \frac{2\pi \text{ rad}}{\text{rev}}$$

$$= 60\pi \text{ rad/s}$$

$$= 188 \text{ rad/s}$$

Angular Acceleration

Recall that the **average linear acceleration** is: $a = (v_f - v_o)/t$

(this is the rate at which the velocity of an object is changing)

We define the angular acceleration (α) of an object as the rate at which its angular velocity changes with time.

$$\text{Average angular acceleration} = \frac{\text{change in angular velocity}}{\text{time taken}}$$

or

$$\alpha = \frac{\omega_f - \omega_o}{t}$$

Worked Example

A wheel starts from rest and attains a rotational velocity of 240 rev/s in 2.0 min. What is its average angular acceleration?

Solution:

We know: $\omega_o = 0$, $\omega_f = 240$ rev/s, $t = 2.0$ min $= 120$ s

therefore: $\alpha = (240 \text{ rev/s} - 0)/120\text{s}$

$= 2.0 \text{ rev/s}^2$

b) What is the wheel's angular speed (in rad/s) 130 s after starting from rest?

$\omega = \alpha t = (2.0 \text{ rev/s}^2)(130 \text{ s}) = 260 \text{ rev/s}$

$= (260 \text{ rev/s})(2\pi \text{ rad/rev}) = 1634 \text{ rad/s}$

Equations for Uniformly Accelerated Angular Motion

There is a great deal of similarity between the linear and angular motion equations.

Linear equation	Corresponding Angular equation
$s = \bar{v}t$	$\theta = \bar{\omega}t$
$v_f = v_o + at$	$\omega_f = \omega_o + \alpha t$
$\bar{v} = \frac{1}{2}(v_f + v_o)$	$\varpi = \frac{1}{2}(\omega_f + \omega_o)$
$v_f^2 = v_o^2 + 2as$	$\omega_f^2 = \omega_o^2 + 2\alpha\theta$
$s = v_o t + \frac{1}{2}at^2$	$\theta = \omega_o t + \frac{1}{2}\alpha t^2$

Worked Example

A wheel turning at 3.0 rev/s coasts to rest uniformly in 18.0 s.

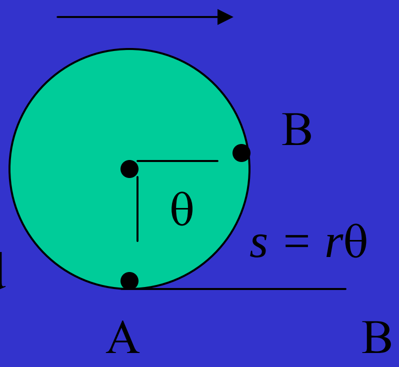
a) What is the deceleration?
b) How many revolutions does it turn while coming to rest?

(-0.167 rev/s^2, 27 rev)

Tangential Quantities

When a spool unwinds a string or a wheel rolls along the ground without slipping, both ***rotational* and *linear*** motions occur.

Consider a wheel with radius r turning.



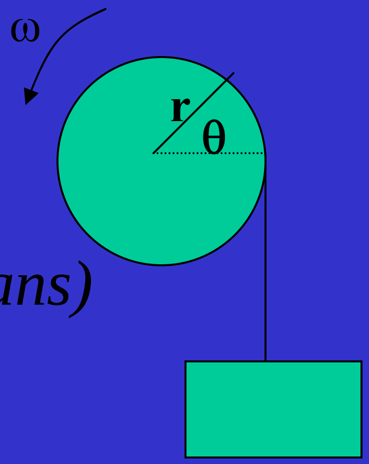
The linear distance the wheel rolls is equal to the tangential distance traveled by a point on the rim.

This allows us to relate linear motion to angular motion for the rolling wheel.

As long as the wheel does not slip, we have $s = r\theta$.

The same goes for a spool and string lifting a weight.

As the spool turns through an angular displacement of θ, a length *s* of string is wound on the spool's rim.



In all such cases, we have:

$$s = r\theta \ (\theta \text{ in radians})$$

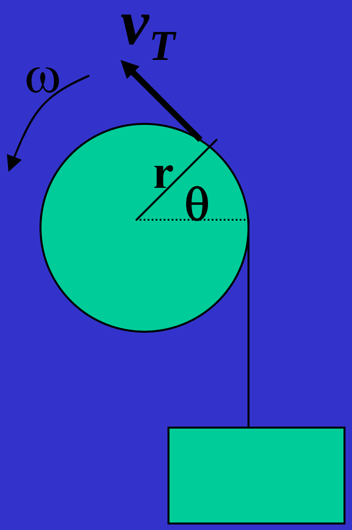
As the spool turns at a certain rate, the mass at the end of the string rises with a certain velocity.

The linear speed of the mass is the same as the speed of a point on the rim of the spool.

The point on the rim is traveling with this speed in a direction always tangent to the spool or wheel.

We call this motion of a point on the rim the tangential velocity, $\boldsymbol{v}_T$, of the point.

Relating the values for ω and *s* together, we get that



Tangential speed = $v_T = \omega r$

and

Tangential acceleration = $a_T = \alpha r$

Where: r = *radius*

α = angular acceleration

ω = rotational velocity

Worked Example

A car with 80 cm diameter wheels starts from rest and accelerates uniformly to 20 m/s in 9.0 s. Find the angular acceleration and final angular velocity of one wheel.

(ans. 5.6 rad/s^2, 50 rad/s)

Worked example

In an experiment with a spool lifting a mass, suppose that the mass starts from rest and accelerates downwards at 8.6 m/s^2. If the radius of the spool is 20 cm, what is its rotation rate at 3.0 s?

(ans. 130 rad/s)

Angular Motion in a Plane – cont'd

Week 9, Lesson 1

- Centripetal Acceleration
- Centripetal Force
- Newton's Law of Gravitation

References/Reading Preparation:

Schaum's Outline Ch. 9

Principles of Physics by Beuche – Ch.7

Centripetal Acceleration

A point mass *m* moving with constant speed *v* around a circle of radius *r* is undergoing acceleration.

This is because although the magnitude of its linear velocity is not changing, the ***direction*** of the velocity is continually changing.

This change in velocity gives rise to an acceleration a_c of the mass – recall that acceleration **ā** = change in velocity / time taken.

This acceleration is directed toward the centre of the circle, and is called the ***centripetal acceleration.*.**

The value of the ***centripetal acceleration*** is given by:

$$a_c = \frac{(\text{tangential speed})^2}{\text{radius of circular path}}$$

$$= \frac{v^2}{r}$$

Where:

v = speed of the mass around the circular path

r = radius of the circular path

Because $v = \omega r$, we also have $a_c = \omega^2 r$

where ω is measured in radians.

Centripetal Force

Newton's first law states that a net force must act on an object if the object is to be deflected from straight-line motion.

Therefore an object traveling on a circular path must have a net force deflecting it from straight-line motion.

For example, a ball being twirled in a circular path is compelled to follow this path by the center-ward pull of the string.

If the string breaks, then the ball will follow a straight line path tangent to the circular path from the point at which it is released.

Now that we know about the centripetal acceleration that acts towards the centre of the circular path, computing the force needed to hold an object of mass *m* in a circular path is a simple task.

We now know that $a_c = v^2/r$ (centripetal acceleration directed towards the centre of the circular path)

Now, a force in the same direction, toward the centre of the circle, must pull on the object to furnish this acceleration.

From the equation $\mathbf{F}_{\text{net}} = m\boldsymbol{a}$, we find this required force.

$$F_c = ma_c = \frac{mv^2}{r}$$

Where *Fc* is called the ***centripetal force***, and is directed toward the centre of the circle.

Worked Example

A 1200 kg car is turning a corner at 8.00 m/s, and it travels along an arc of a circle in the process.

a) If the radius of the circle is 9.00 m, how large a horizontal force must the pavement exert on the tires to hold the car in a circular path?
b) What minimum coefficient of friction must exist in order for the car not to slip?

(ans. 8530 N, 0.725)

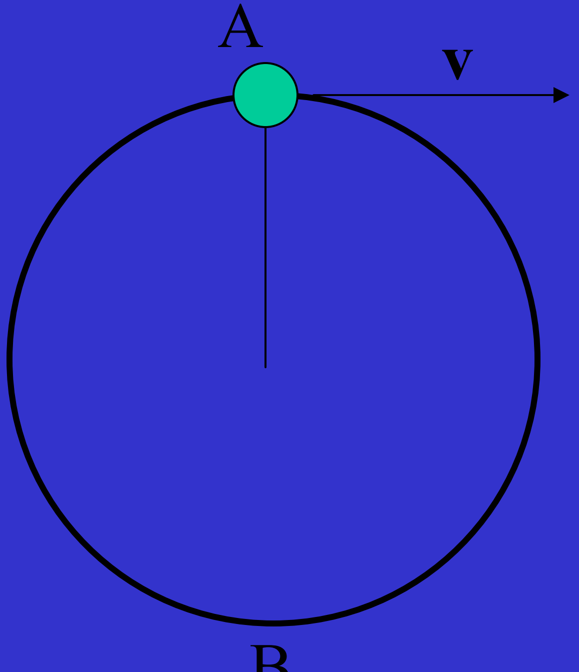
Worked Example

A mass of 1.5 kg moves in a circle of radius 25 cm at 2 rev/s. Calculate a) the tangential velocity, b) the centripetal acceleration, and c) the required centripetal force for the motion.

(ans. 3.14 m/s, 39.4 m/s^2 radially inward, 59N)

Worked Example

A ball tied to the end of a string is swung in a vertical circle of radius *r*. What is the tension in the string when the ball is at point A if the ball's speed is *v* at that point? (Do not neglect gravity). What would the tension in the string be at the bottom of the circle if the ball was going speed *v*?



Worked Example

A curve in a road has a 60 m radius. It is to be banked so that no friction force is required for a car going at 25 m/s to safely make the curve. At what angle should it be banked?

(ans. 47°)

Newton's Law of Gravitation

Two uniform spheres with masses m_1 and m_2 that have a distance r between centres attract each other with a radial force of magnitude:

$$F = G\,\frac{m_1 m_2}{r^2}$$

where G = gravitational constant = $6.672 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$

Illustration

Two uniform spheres, both of 70.0 kg mass, hang as pendulums so that their centres of mass are 2.00 m apart. Find the gravitational force of attraction between them.

(ans. 8.17×10^{-8} N)

Example

A spaceship orbits the moon at a height of 20,000 m. Assuming it to be subject only to the gravitational pull of the moon, find its speed and the time it takes for one orbit. For the moon, $m_m = 7.34 \times 10^{22}$ kg, and $r = 1.738 \times 10^6$ m.

(ans. 1.67 km/s, 110 min)

Rigid Body Rotation

Week 9, Lesson 2

- Rotational Work & Kinetic Energy
- Rotational Inertia

References/Reading Preparation:
Schaum's Outline Ch. 10
Principles of Physics by Beuche – Ch.8

Rotational Work and Kinetic Energy

It is easy to see that a rotating object has kinetic energy.

Recall that for *linear motion*: KE = $\frac{1}{2}mv^2$

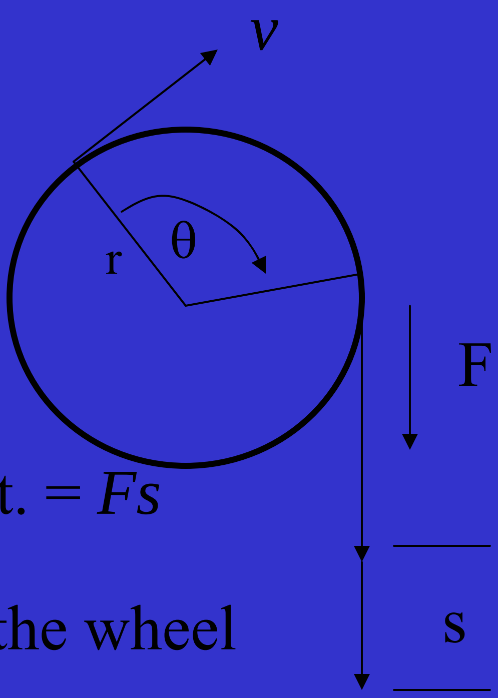
(we call this the ***translational kinetic energy – KE_t***)

When we consider that a rotating object is made up of many tiny bits of mass, each moving as the object turns, we can see that each tiny mass has a mass and a velocity.

Therefore, each mass, has kinetic energy of $\frac{1}{2}m_1v_1^2$

Consider this wheel with a string attached:

When a force F pulls on the string, the wheel begins to rotate.



The work done by the force as it pulls the string is:

Work done by F = Force x dist. = Fs

As the length (s) of string is unwound, the wheel turns through an angle θ.

Recall that $s = r\theta$. Therefore, work done by F = $Fs = Fr\theta$

The term Fr *(force x dist.)* is the torque, τ.

Therefore: $W = \tau\theta$

According to the work-energy theorem, the work done on the wheel by the applied torque *must appear* as Kinetic Energy, KE.

We call the Kinetic Energy resident in a rotating object the Kinetic Energy of Rotation, and is designated as:

$$KE_r$$

If we look at a wheel made up of many tiny masses, undergoing a velocity, then each mass has KE. The KE of the wheel is then the sum of the kinetic energies of all of the tiny masses.

KE of the wheel = $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots + \frac{1}{2}m_nv_n^2$

Each tiny mass travels around a circle with radius r_{n}.

For a mass m_1, with velocity v_1, its angular velocity is related to the tangential velocity by $v_1 = \omega r_1$

Therefore, $\frac{1}{2}\, \mathrm{m}_1\mathrm{v}_1^2 = \frac{1}{2}\, \mathrm{m}_1\omega^2 r_1^2$

KE of the wheel = $\frac{1}{2}m_1\omega^2 r_1^2 + \frac{1}{2}m_2\omega^2 r_2^2 + \ldots + \frac{1}{2}m_n\omega^2 r_n^2$

KE of the wheel = $\frac{1}{2}m_1\omega^2 r_1^2 + \frac{1}{2}m_2\omega^2 r_2^2 + \ldots + \frac{1}{2}m_n\omega^2 r_n^2$

Taking out common factors,

KE of the wheel = $\frac{1}{2}\omega^2 (m_1 r_1^2 + m_2 r_2^2 + \ldots m_n r_n^2)$

The term in the brackets is called the **moment of inertia (I)** of the rotating object.

Therefore, $KE_r = \frac{1}{2} I\omega^2$ (rotational kinetic energy)

If we apply a torque (τ) to a rotating wheel;

$\tau = I\alpha$ Where α is in rad/s.

Rotational Inertia

We know that rotating objects have *inertia*.

When we turn off a fan, the blade coasts for some time as the friction forces of the air and the axle bearings slowly cause it to stop.

The moment of inertia, I, of the fan blade measures its rotational inertia.

We can understand this as follows

In linear motion, the inertia of an object is represented by the object's mass.

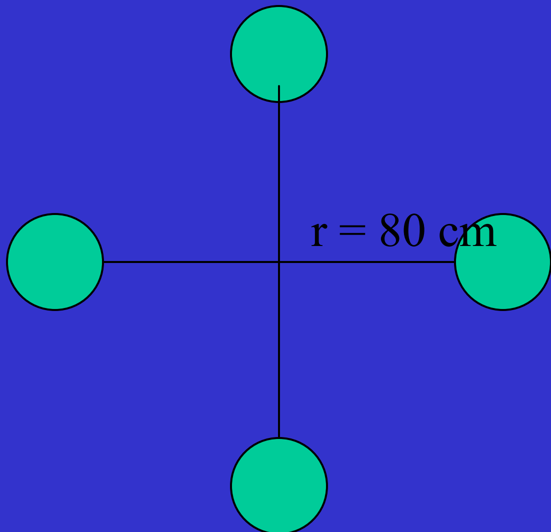
From $F = ma$, we have: $m = F/a$

If we fix 'a' at an acceleration of 1 m/s^2, then this equation shows us that the mass tells us how large a force is required to produce the given acceleration.

larger mass, then, larger force

smaller mass, then, smaller force

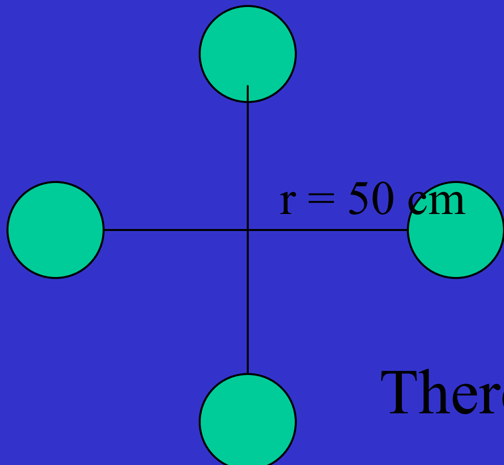
Let's look at these two wheels, each having equal masses of 3 kg.



$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots = \Sigma m_i r_i^2$$

$$I_A = 3(0.8)^2 + 3(0.8)^2 + 3(0.8)^2 + 3(0.8)^2$$
$$= 7.68\ \mathrm{kg{\cdot}m^2}$$

$$I_B = 3(0.5)^2 + 3(0.5)^2 + 3(0.5)^2 + 3(0.5)^2$$
$$= 3.00\ \mathrm{kg{\cdot}m^2}$$



Note that (B) has a smaller moment of inertia Their masses are the same. The I's differ because the masses are further from the axis.

Therefore a greater torque is needed in A than in B.

The moment of inertia for any object s calculated by dividing the object into tiny masses and using using calculus.

The moments of inertia (about an axis through the centre of mass) of several uniform objects are shown in your text.

In all cases, I is the product of the object's mass and the square of a length.

Generally we can write the equation in the form $I = Mk^2$

Where M is the total mass of the object, and
k is the *radius f gyration*, and is the distance a point mass M must be from the axis if the point mass is to have the same I as the object.

Summary

1) An object of mass M possesses rotational inertia, where,

$$I = Mk^2$$

2) A rotating object has rotational kinetic energy, where,

$$\mathrm{KE}_r = \tfrac{1}{2} I\omega^2$$

3) A torque (τ) applied to an object that is free to rotate gives the object an angular acceleration, where,

$$\tau = I\alpha$$

4) The work done by a torque, τ, when it acts through an angle θ is $\tau\theta$.

Rigid Body Rotation (cont'd)

Week 10, Lesson 1

- Parallel-Axis Theorem
- Combined Rotation & Translation
- Angular Momentum

References/Reading Preparation:
Schaum's Outline Ch. 10
Principles of Physics by Beuche – Ch.8

Summary From Last Lecture

1) An object of mass M possesses rotational inertia, where,

$$I = Mk^2$$

2) A rotating object has rotational kinetic energy, where,

$$KE_r = \tfrac{1}{2} I\omega^2$$

3) A torque (τ) applied to an object that is free to rotate gives the object an angular acceleration, where,

$$\tau = I\alpha$$

4) The work done by a torque, τ, when it acts through an angle θ is $\tau\theta$.

Parallel-Axis Theorem

The moments of inertia of the objects shown in your text are calculated about the centres of the mass of the objects.

There s a very simple and useful theorem by which we can calculate the moments of inertia of these same objects about *any other axis* which is parallel to the centre of mass axis.

The moment of inertia of an object about an axis O which is parallel to the centre of mass of the object is:

$$I = I_c + Mh^2$$

Where, I_c = moment of inertia about an axis through the mass centre

M = total mass of the body

h = perpendicular distance between the two parallel axes

Worked Example

Determine the moment of inertia of a solid disk of radius r and mass M about an axis running through a point on its rim and perpendicular to the plane of the disk.

(ans. 3/2 Mr^2)

Worked Example

Find the rotational kinetic energy of the earth due to its daily rotation on its axis. Assume a uniform sphere of
$M = 5.98 \times 10^{24}$ kg, $r = 6.37 \times 10^{6}$ m

Worked Example

A certain wheel with a radius of 40 cm has a mass of 30 kg and a radius of gyration, *k*, of 25 cm. A cord wound around its rim supplies a tangential force of 1.8 N to the wheel which turns freely on its axis. Find the angular acceleration of the wheel.

(ans. $\alpha = 0.384$ rad/s^2)

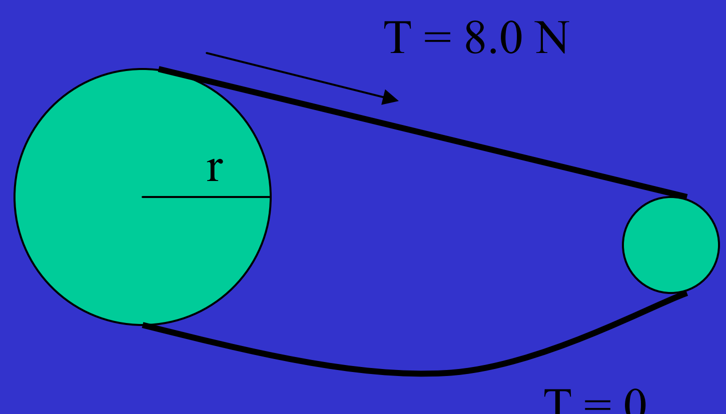
Worked Example

The larger wheel shown has a mass of 80 kg and a radius *r* of 25 cm. It is driven by a belt as shown. The tension in the upper part of the belt is 8.0 N and that for the lower part is essentially zero. Assume the wheel to be a uniform disk.

a) How long does it take for the belt to accelerate the larger wheel from rest to a speed of 2.0 rev/s?

b) How far does the wheel turn in this time (i.e., what is the angular displacement, θ)?

c) What is the rotational KE?



(ans. $t = 15.7$ s
$\theta = 98.6$ rad
$KE_r = 197$ J)

Worked Example

A 500 g uniform sphere of 7.0 cm radius spins at 30 rev/s on an axis through its centre. Find its:

a) KE_r (ans. 17.3 J)

b) Angular momentum (ans. 0.184 $kg \cdot m^2/s$)

c) Radius of gyration (ans. 0.0443 m)

Combined Rotation and Translation

The kinetic energy, KE, of a rolling ball or other rolling object of mass *M* is the sum of:

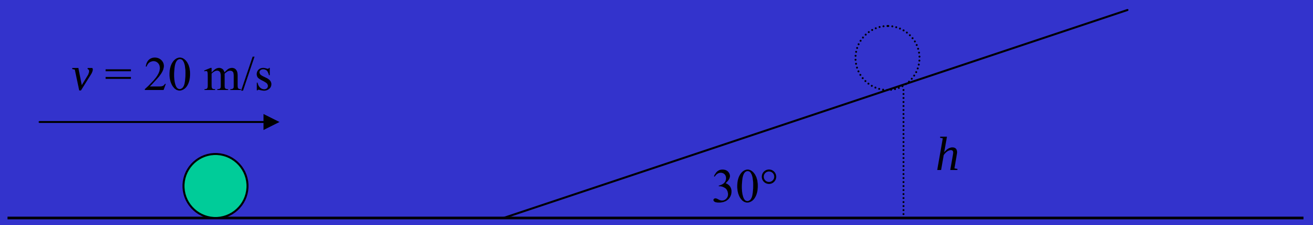
1) Its rotational KE *about an axis through its centre of mass*, and
2) The translational KE of an equivalent point mass moving with the centre of mass.

$$\text{KE total} = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}Mv^2$$

Note that I is the moment of inertia of the object about an axis through its mass centre.

Worked Example

As shown, a uniform sphere rolls on a horizontal surface at 20 m/s and then rolls up the incline. If friction losses are negligible, what will be the value of *h* where the ball stops?



(ans. h = 28.6 m)

Angular Momentum

Rotational, or angular, momentum is associated with the fact that a rotating object persists in rotating.

Angular momentum is a vector quantity with magnitude $I\omega$ and is directed along the axis of rotation.

If the net torque on a body is zero, its angular momentum will Remain unchanged in both magnitude and direction. This is the *Law of conservation of angular momentum.*

Worked Example

A disk of moment of inertia I_1 is rotating freely with angular speed ω_1 when a second, non-rotating, disk with moment of inertia I_2 is dropped on it. The two then rotate as a unit. Find the angular speed.

ans. $\omega = (I_1\omega_1)/(I_1 + I_2)$